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One-particle equal time correlation function for the spin-incoherent infinite U Hubbard chain

Vadim V Cheianov[1] **and M B Zvonarev**[2]

[1] Physics Department, Lancaster University, Lancaster, LA1 4YB, UK
[2] DPMC-MaNEP, University of Geneva, 24 quai Ernest-Ansermet, 1211 Geneva 4, Switzerland

E-mail: v.cheianov@gmail.com and mikhail.zvonarev@physics.unige.ch

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Abstract
We present a calculation of the one-particle equal time correlation function $\rho(x)$ for the one-dimensional (1D) Hubbard model in the infinite U limit. We consider the zero temperature spin incoherent regime, which is obtained by first taking the limit $U \to \infty$ and then the limit $T \to 0$. Using the determinant representation for $\rho(x)$, we derive analytical expressions for both large and small x at an arbitrary filling factor $0 < \varrho < 1/2$. The large x asymptotics of $\rho(x)$ is found to be remarkably accurate starting from $x \sin(2\pi\varrho) \sim 1$. We find that the one-particle momentum distribution function $\rho(k)$ is a smooth function of k, and $\rho'(k)$ is peaked at $k = 2k_F$ in contrast to spin-coherent liquid obeying the Luttinger theorem.

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Recent progress in the calculation of the dynamical correlation functions in the spin-incoherent gas of impenetrable spin 1/2 fermions [1–3] have attracted a lot of interest [4]. It was found that the infrared asymptotic behavior of the correlation functions, although consistent with the assumption of spin–charge separation, is not adequately described by the Luttinger model. This is to be contrasted with the asymptotic behavior of the previously studied correlation functions of the infinite U Hubbard model in the 'antiferromagnetic' ground state, understood as a limit of the ground state of the Hubbard model as $U \to \infty$. In the latter case, the Luttinger model gives correct predictions [5–7].

In this paper we investigate the spin-incoherent regime in the 1D Hubbard model [8]

$$H = -\sum_{x;\sigma} \left(\psi^\dagger_{x,\sigma}\psi_{x+1,\sigma} + \text{h.c.}\right) + \sum_x U n_{x,\uparrow} n_{x,\downarrow}, \tag{1}$$

in the limit $U \to \infty$. Here $\psi_{x,\sigma}$ are fermion fields with the spin index $\sigma = \uparrow, \downarrow$, and $n_{x,\sigma} = \psi^\dagger_{x,\sigma}\psi_{x,\sigma}$ are the local fermion number operators. We will concentrate on the one particle equal time

$$\rho(x) = \langle \psi^\dagger_{x,\uparrow}\psi_{0,\uparrow}\rangle, \tag{2}$$

at an arbitrary fixed filling factor

$$\varrho = \tfrac{1}{2}\langle n_{x,\uparrow} + n_{x,\downarrow}\rangle \tag{3}$$

and in the limit $T \to 0$.

The ground state of the model (1) at infinite U is infinitely degenerate with respect to local spin rotations [5]. Since the limit $T \to 0$ is taken after the limit $U \to \infty$, the thermal average $\langle\rangle$ in (2) reduces to the average over the infinitely degenerate ground state. This is what we call an average taken in the zero temperature spin-incoherent regime of the model.

Recently, the determinant representation for the dynamical correlation functions of the infinite U Hubbard model (1) in the spin-incoherent regime was obtained [9]. For the equal time correlation function (2), the determinant representation, given in [9], can be written in the following form:

$$\rho(x) = \frac{1}{8\pi \mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{z}F(z)B_{--}(z)\det(\hat{I}+\hat{V})(z). \tag{4}$$

Here the function $F(z)$ is

$$F(z) = 1 + \frac{z}{2-z} + \frac{1}{2z-1}. \tag{5}$$

The determinant

$$\det(\hat{I}+\hat{V}) = \sum_{N=0}^{\infty}\frac{1}{N!}\int_{-K}^{K}\mathrm{d}k_1\ldots\int_{-K}^{K}\mathrm{d}k_N\begin{vmatrix} V(k_1,k_1) & \cdots & V(k_1,k_N)\\ \vdots & \ddots & \vdots\\ V(k_N,k_1) & \cdots & V(k_N,k_N)\end{vmatrix} \tag{6}$$

is the Fredholm determinant of a linear integral operator $\hat{V}$ with the kernel

$$V(k,p) = \frac{e_+(k)e_-(p) - e_+(p)e_-(k)}{2\tan\left[\frac{1}{2}(k-p)\right]} \tag{7}$$

defined on $[-K, K] \times [-K, K]$. Here

$$K = 2\pi\varrho \tag{8}$$

is twice the Fermi momentum. The functions $e_\pm$ entering equation (7) are defined as follows

$$e_-(k) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\mathrm{i}kx/2}, \tag{9}$$

$$e_+(k) = \frac{\mathrm{i}}{2}\frac{1}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kx/2}(1-z). \tag{10}$$

The function $B_{--}(z)$ is

$$B_{--}(z) = \int_{-K}^{K}\mathrm{d}k\, e_-(k)(\hat{I}+\hat{V})^{-1}e_-(k). \tag{11}$$

Consider the contour integral in equation (4). According to equations (7) and (10), the Fredholm operator $\hat{V}$ is linear in z. This implies [2] that the product $B_{--}(z)\det(\hat{I}-\hat{V})(z)$ is analytic in the complex z-plane. Therefore, the integral is given by the residue of the integrand at the pole $z = 1/2$ of the function $F(z)$

$$\rho(x) = \tfrac{1}{4}B_{--}(1/2)\det(\hat{I}+\hat{V})(1/2). \tag{12}$$

Consider the *short distance* behavior of $\rho(x)$ first. For any x the kernel (7) can be written as a sum of $2x$ separable kernels (recall that x is a discrete variable, $x = 0, 1, 2, \ldots$)

$$V(k,p) = \frac{z-1}{4\pi}\sum_{m=1}^{2x}u_m(k)u_m^*(p), \tag{13}$$

where

$$u_m(k) = \begin{cases} e^{i(m-\frac{x}{2})k}, & m = 1, \ldots, x \\ e^{-i(m-\frac{3x}{2})}, & m = x+1, \ldots, 2x \end{cases}. \tag{14}$$

Therefore, $\det(\hat{I} + \hat{V})$ can be expressed in terms of the determinant

$$\det(\hat{I} + \hat{V}) = \det_{2x}(\mathbf{I} + \mathbf{V}) \tag{15}$$

of an $2x \times 2x$ matrix $\mathbf{V}$:

$$\mathbf{V} = \frac{z-1}{2\pi} \left(\begin{array}{c|c} Q & P \\ P & Q \end{array} \right). \tag{16}$$

Here Q and P are the $x \times x$ matrices with the entries defined by

$$Q_{mn} = \frac{\sin[K(m-n)]}{m-n}, \qquad n, m = 1, \ldots, x \tag{17}$$

$$P_{mn} = \frac{\sin[K(m+n-x)]}{m+n-x}, \qquad n, m = 1, \ldots, x \tag{18}$$

where

$$Q_{nn} = P_{(x-n)n} = K. \tag{19}$$

For $B_{--}(z)$ one has

$$B_{--} = \frac{2\sin Kx}{\pi x} - \frac{z-1}{4\pi} \mathbf{a}^{\mathrm{T}} (\mathbf{I} + \mathbf{V})^{-1} \mathbf{b}, \tag{20}$$

where the $2x$-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ are defined by

$$\mathbf{a}_n = \begin{cases} \dfrac{2\sin Kn}{\sqrt{\pi} n}, & n = 1, \ldots, x \\ \dfrac{2\sin K(n-2x)}{\sqrt{\pi}(n-2x)}, & n = x+1, \ldots, 2x \end{cases} \tag{21}$$

and

$$\mathbf{b}_n = \frac{2\sin K(n-x)}{\sqrt{\pi}(n-x)}, \qquad n = 1, \ldots, 2x. \tag{22}$$

Equations (15) through (22) combined with (12) are convenient for the calculation of $\rho(x)$ at small enough x. For example,

$$\rho(0) = \frac{K}{2\pi}, \tag{23}$$

$$\rho(1) = \frac{\sin K}{2\pi}, \tag{24}$$

$$\rho(2) = \frac{\sin^2 K}{4\pi^2} + \frac{(2\pi - K)\sin 2K}{8\pi^2}. \tag{25}$$

With increasing x the complexity of the exact expression for $\rho(x)$ grows rapidly.

Next, we calculate the *long distance* asymptotics of equation (2) using the determinant representation (4). Technically, the asymptotic analysis will be similar to that carried out for the continuous limit of the model in [2].

To calculate $\det(\hat{I} + \hat{V})$, we write the difference equation for the kernel (7):

$$V(k, p; x+1) = e^{\frac{i}{2}(k-p)} V(k, p; x) + i e_-(k; x) e_+(p; x) \cos\frac{k-p}{2}. \tag{26}$$

From this equation it follows that

$$\det(\hat{I}+\hat{V})(x+1;z)=\det(\hat{I}+\hat{V})(x;z)W(x;z), \tag{27}$$

where

$$W(x)=\det\begin{bmatrix}1+\frac{\mathrm{i}}{2}B_{+-}(x) & \frac{\mathrm{i}}{2}D_{-+}(x)\\ \frac{\mathrm{i}}{2}C_{+-}(x) & 1+\frac{\mathrm{i}}{2}A_{-+}(x)\end{bmatrix} \tag{28}$$

and

$$A_{ab}=\int_{-K}^{K}\mathrm{d}k\, e_a(k)\,\mathrm{e}^{-\mathrm{i}k}(\hat{I}+\hat{V})^{-1}[\mathrm{e}^{-\mathrm{i}k}e_b(k)], \tag{29}$$

$$B_{ab}=\int_{-K}^{K}\mathrm{d}k\, e_a(k)(\hat{I}+\hat{V})^{-1}e_b(k), \tag{30}$$

$$C_{ab}=\int_{-K}^{K}\mathrm{d}k\, \mathrm{e}_a(k)\,\mathrm{e}^{-\mathrm{i}k}(\hat{I}+\hat{V})^{-1}e_b(k), \tag{31}$$

$$D_{ab}=\int_{-K}^{K}\mathrm{d}k\, e_a(k)\,\mathrm{e}^{-\mathrm{i}k}(\hat{I}+\hat{V})^{-1}e_b(k). \tag{32}$$

The indices a and b run through two values: $a,b=\pm$.

The resolvent operator $(\hat{I}+\hat{V})^{-1}$ and, therefore, the functions (29)–(32) can be found from the solution of the corresponding matrix Riemann–Hilbert problem [10]. The scheme of the asymptotic solution of the matrix Riemann–Hilbert problem associated with the kernel (7) is very similar to that given in [2]. It is based on the nonlinear steepest-descend method [11]. The main results of the asymptotic analysis are as follows. For $z=1/2$

$$W(x;z=1/2)=2^{-K/\pi}\left[1+\frac{\nu^2}{2x}\right]+\delta W(x), \tag{33}$$

where

$$\nu=\frac{\ln 2}{\pi}. \tag{34}$$

The residual term $\delta W(x)$ decays as x^{-2} for $x\sin K\gg 1$. Solving equation (27) with W given by equation (33) one gets in the large x limit

$$\det(\hat{I}+\hat{V})(x)=\mathrm{e}^{C(K)}(\sin K)^{\frac{\nu^2}{2}}\cdot 2^{-\frac{K}{\pi}x}x^{\frac{\nu^2}{2}}, \tag{35}$$

where $C(K)$ is independent of x. Numerically, $\exp[C(K)]$ is close to unity for all K, as can be seen in figure 1. The exact expression for $C(0)$ is given in [2] and is, numerically, equal to $0.0550839\ldots$. This agrees perfectly with figure 1.

The asymptotic formula for the one particle correlation function (4) reads

$$\rho(x)=\frac{\mathrm{i}\pi 2\sqrt{2}\,\mathrm{e}^{C(K)}(\sin K)^{\frac{\nu^2}{2}}}{\cosh^2(K\nu/2)}\mathrm{e}^{-\nu Kx}x^{\frac{\nu^2}{2}}\left[\frac{(2\sin K)^{-\mathrm{i}\nu}}{\Gamma(-\mathrm{i}\nu/2)^2}\frac{\mathrm{e}^{\mathrm{i}Kx}}{x^{1+\mathrm{i}\nu}}-\frac{(2\sin K)^{\mathrm{i}\nu}}{\Gamma(\mathrm{i}\nu/2)^2}\frac{\mathrm{e}^{-\mathrm{i}Kx}}{x^{1-\mathrm{i}\nu}}\right] \tag{36}$$

with the relative correction of the order of x^{-1}. The formula (36) is the main result of the paper.

Let us discuss equation (36). The structure of the correlation function is essentially the same as for the impenetrable fermion gas [1, 2, 12]. The correlation function contains the exponentially decaying factor $\exp(-\nu Kx)$, and factors obeying the power law scaling. The complex-valued anomalous exponents do not depend on K or, equivalently, on the filling

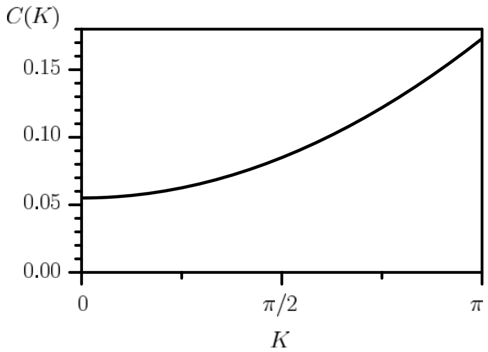

Figure 1. The constant $C(K)$ in equations (35) and (36) plotted as a function of K. The plot data were obtained from the comparison of the asymptotic formula (35) with the numerically calculated Fredholm determinant (6).

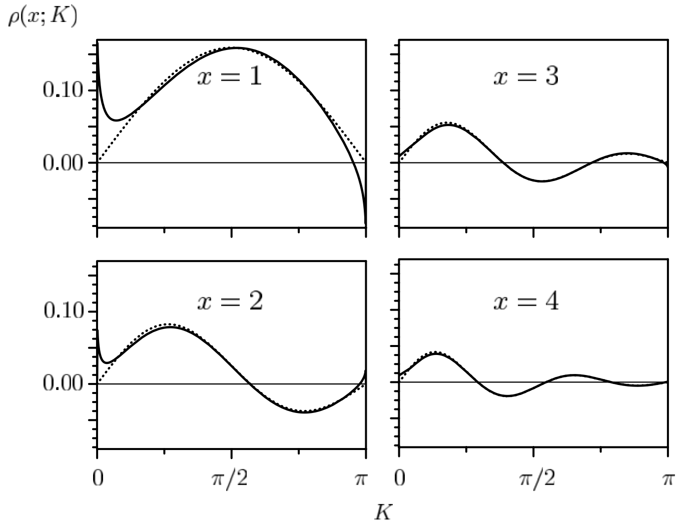

Figure 2. Correlation function $\rho(x)$ plotted as a function of K for $x = 1, \ldots, 4$. The asymptotic result (36) (solid line) is in good agreement with the exact result (dotted line) even for small $x \sin K$.

factor ϱ. A similar situation takes place for the infinite U Hubbard model in the Luttinger regime: the Luttinger scaling exponents do not depend on the filling factor [7]. The results for the continuous model, impenetrable fermion gas [1, 2, 12], can be recovered by taking the limit $K \to 0$ in equation (36) at a fixed Kx.

Formally, the asymptotic formula given in equation (36) is valid for $x \sin K \gg 1$. Nevertheless, it is remarkably good even for $x \sin K \sim 1$ as can be seen from figure 2, where the exact expression obtained from equations (15) through (22) is compared with the asymptotics equation (36).

Finally, consider the momentum distribution function

$$\rho(k) = \sum_{x=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} \rho(x). \tag{37}$$

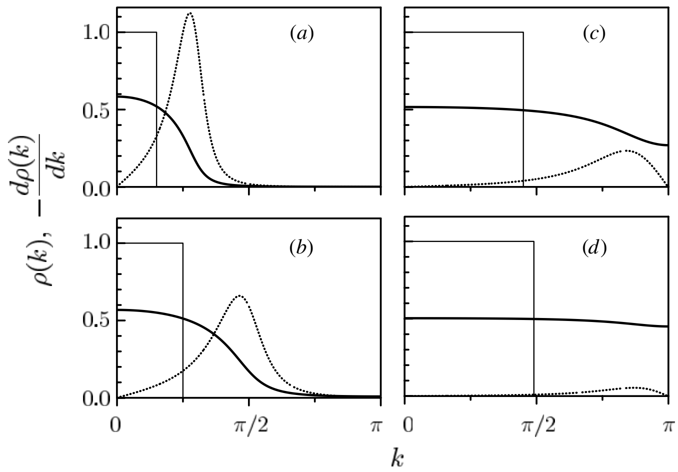

Figure 3. Momentum distribution function $\rho(k)$ (thick line) and its derivative $-\mathrm{d}\rho(k)/\mathrm{d}k$ (dotted line) for different filling factors ϱ : (a) $\varrho = 1/8$; (b) $\varrho = 1/4$; (c) $\varrho = 0.45$; and (d) $\varrho = 0.49$. The Fermi–Dirac distribution (thin line) corresponding to these filling factors is shown for comparison. The function $\rho(k)$ satisfies $\rho(k) = \rho(-k)$.

Due to the exponentially decaying term in the asymptotic expression (36), the function $\rho(k)$ is continuous with all its derivatives for all k. Combining the short distance representation (15)–(22) and the long distance expansion (36) we plot $\rho(k)$ for $0 \leqslant k \leqslant \pi$ at different filling factors ϱ in figure 3. Note that the smoothness of $\rho(k)$ in the spin-incoherent regime, considered here, is in contrast with the Luttinger regime considered in [5], where $\mathrm{d}\rho(k)/\mathrm{d}k$ is singular at the Fermi momentum $k_F = K/2$, in accordance with the Luttinger theorem [13]. Another peculiarity of the spin-incoherent regime, seen in figure 3, is that $d\rho(k)/dk$ is peaked around $k = 2k_F$ rather than at $k = k_F$ predicted by the Luttinger theorem.

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